Chebyshev Polynomials as Extremal Polynomials with Computer Algebra*

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Abstract

Different approaches are known for the introduction of the classical Chebyshev polynomials. Although mostly they are introduced in connection with trigonometric identities or defined as orthogonal polynomials or a solution of a particular differential equation, we take extremality as a defining property. We discuss the current computer tools which make the exploration of extremal polynomials possibly more easy and enjoyable. Finally we investigate ways to generalize the notion of classical Chebyshev polynomials as extremal polynomials which are less known. The author used the general computer algebra system Mathematica for the development, but other systems can be also suitbale for the exploration: a part of a GeoGebra book as an illustrative example is also included.

1 Introduction

The expressions manipulated by the general purpose computer algebra systems (CAS) typically include polynomials in several variables. Powerful algorithms in symbolic computation such as Groebner basis, cylindrical algebraic decomposition, real and complex quantifier elimination make it possible to investigate nontrivial computational problems that can be expressed by polynomials. Some typical problem class is polynomial optimization, polynomial approximation, robot motion planing, see [4, p. 2]. Therefore it is reasonable to expect that CAS also provides a handy environment for exploring classical Chebyshev polynomials of the first kind. They may pop up in different contexts and courses at university level, but some relations and properties can be treated already in the secondary level. In this short paper, without attempting to be comprehensive in this huge field, we highlight an approach which emphasizes a *defining extremal property* of these objects. We investigate and demonstrate the level of the algorithmic, computational support that a user/learner may get through

*Paper based on the talk presented at CADGME 2014, Halle, Germany.

the different exploration paths. This is critical due to the fact that computational complexity can grow rapidly with the growth of the degree of the polynomials investigated. Finally we discuss how the problems and the notions can be generalized. Before we turn into extremal polynomials, as a contrast, we briefly recall two other viable alternative approaches for the introduction of Chebyshev polynomials. The first one is suitable even in the secondary level with basic trigonometric background knowledge. The second approach, which typically occurs in undergraduate courses in numerics, treats them as members of an orthogonal polynomial system.

1.1 Approach 1: A trigonometric identity

Problem 1 *Find an explicit expression for* $\cos nx$ *in terms of* $\cos x$*, that is, find* T_n *such that* $\cos nx = T_n(\cos x)$.

The problem is trivial for $n = 0, 1$ (i.e., $T_0(x) = 1, T_1(x) = x$) and also easy for $n = 2$ knowing a standard trigonometric identity for the double angle:

$$T_2(\cos x) = \cos(2x) = 2(\cos x)^2 - 1 \Rightarrow T_2(x) = 2x^2 - 1. \tag{1}$$

Here the computer algebra system such as Mathematica[1] [15] can help verify or generate the sequence of the polynomials as follows:

$$\begin{aligned}&\mathbf{Table}[\\&\mathbf{Expand}[\mathbf{TrigExpand}[\mathbf{Cos}[\mathbf{nx}]]/.\mathbf{Sin}[\mathbf{x}] \to \mathbf{Sqrt}[\mathbf{1}-\mathbf{Cos}[\mathbf{x}]^{\mathbf{2}}]/.\mathbf{Cos}[\mathbf{x}] \to \mathbf{x}],\\&\{\mathbf{n}, \mathbf{5}\}],\end{aligned} \tag{2}$$

to get the finite list of polynomials:

$$\{1, -1 + 2x^2, -3x + 4x^3, 1 - 8x^2 + 8x^4, 5x - 20x^3 + 16x^5\}. \tag{3}$$

The polynomials in the list above, i.e., $T_n(x) \quad (n = 0, 1, \dots)$ is called the *n-th Chebyshev polynomial* (of the first kind) and will be the main player in this article. Root distribution, minimum and maximum points, etc. can be investigated now. For example, on the unit interval $\mathbf{I} = [-1, 1]$,

$$\min_{x\in\mathbf{I}} T_n(x) = -1, \quad \max_{x\in\mathbf{I}} T_n(x) = 1 \quad (n > 0), \tag{4}$$

and all the roots are simple and in $\mathbf{I}$. Here we can also exploit the graphical tools of a CAS at our disposal to depict the graphs of the functions obtained.

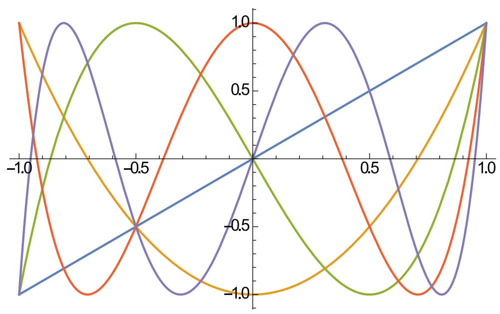

Figure 1: Graphs of the Chebyshev polynomials T_n in $\mathbf{I}$

[1] Sample Mathematica codes are boldfaced in the text

Finally, our students can be asked for a recursive relation which can be deduced by another trigonometric identity

$$T_0(x) = 1, T_1(x) = x; T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x) \quad (n > 1). \tag{5}$$

1.2 Approach 2: Orthogonal polynomial system

Problem 2 *Find a set of orthogonal polynomials $\{O_n\}$ in the unit interval $[-1, 1]$ w.r.t. the weight function $\rho(x) = \frac{1}{\sqrt{1-x^2}}$.*

Assume that $O_0 = 1, O_1(x) = x$ is already known and we are looking for the monic quadratic $a_0 + a_1x + x^2$ which is orthogonal to O_0 and O_1. In this context two functions f and g is orthogonal iff

$$\langle f, g\rangle := \int_{-1}^{1} \frac{f(x)g(x)}{\sqrt{1-x^2}} = 0. \tag{6}$$

The two inner products $\langle O_0, O_2\rangle$ and $\langle O_1, O_2\rangle$ are computed with Mathematica as follows.

$$\mathbf{Integrate}[(\mathbf{x^2} + \mathbf{a1x} + \mathbf{a0})/\mathbf{Sqrt}[\mathbf{1} - \mathbf{x^2}], \{\mathbf{x}, -\mathbf{1}, \mathbf{1}\}], \tag{7}$$

$$\mathbf{Integrate}[\mathbf{x}(\mathbf{x^2} + \mathbf{a1x} + \mathbf{a0})/\mathbf{Sqrt}[\mathbf{1} - \mathbf{x^2}], \{\mathbf{x}, -\mathbf{1}, \mathbf{1}\}]. \tag{8}$$

What we get is $(1/2 + a_0)\pi$ and $(\pi/2)a_1$. It is obvious the two inner products are zero iff $a_0 = -\frac{1}{2}$ and $a_1 = 0$, thus

$$O_2(x) = x^2 - \frac{1}{2}. \tag{9}$$

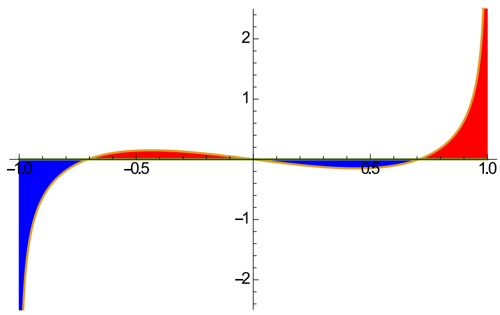

Figure 2: Visual demonstration of the orthogonality of O_1 and O_2

Observe that although O_2 is not same polynomial as T_2, it differs from T_2 only by a constant factor! A Gram-Schmidt type orthogonalization process can generate recursively all the elements of the polynomial system O_j as follows (although it may take some time to compute O_n for bigger n's using the implementation below):

$$\mathbf{P}[\mathbf{0}] = \mathbf{1};$$
$$\mathbf{P}[\mathbf{n\_}] := \mathbf{x^n} - \mathbf{Sum}[\mathbf{Integrate}[\mathbf{x^n P}[\mathbf{j}]/\mathbf{Sqrt}[\mathbf{1} - \mathbf{x^2}], \{\mathbf{x}, -\mathbf{1}, \mathbf{1}\}]$$
$$/\mathbf{Integrate}[\mathbf{P}[\mathbf{j}]\mathbf{P}[\mathbf{j}]/\mathbf{Sqrt}[\mathbf{1} - \mathbf{x^2}], \{\mathbf{x}, -\mathbf{1}, \mathbf{1}\}]\mathbf{P}[\mathbf{j}], \{\mathbf{j}, \mathbf{0}, \mathbf{n} - \mathbf{1}\}]$$

What we get finally for $n \leq 5$ is

$$O_1(x) = x, O_2(x) = x^2–1/2, O_3(x) = x^3–3/4x, O_4 = x^4–x^2+1/8, O_5(x) = x^5–5/4x^3+5x/16. \quad (10)$$

We see that the two polynomial systems can be linked via the the factor 2^{n-1}, i.e.,

$$T_n(x) = 2^{n-1}O_n(x) \quad n = 1, 2, \ldots . \quad (11)$$

2 Chebyshev polynomials as extremal polynomials

In this section we now turn to our main target, to extremal polynomials. Extremal objects are very important in mathematics. Extremal algebraic and trigonometric polynomials are basic to approximation theory and extremal graph theory is an estalished subfield of graph theory. Extremal objects have usually nice properties and can play also important role in estimations and problem solving strategies. Similar to Section 1, two types of problems are investigated.

2.1 Approach 3: Extremal polynomials in the unit interval

Problem 3 *Find a polynomial among the monic polynomials of degree n with real coefficients which deviates least from the constant zero polynomial in* $\mathbf{I} = [-1, 1]$.

Again, we start to investigate the degree two case: First we use the symbolic optimization tools of Mathematica as a black-box [3]. Roughly, the built-in functions **Minimize** and **Maximize** are able to deal with constraint symbolic (polynomial) optimization problems and return a extremal value and an extremal point. Note that the square of the deviation is investigated. After the call

$$\mathbf{Minimize[Maximize[\{((x^2 + a1x + a0) - 0)^2, -1 \leq x \leq 1\}, x][[1]], \{a0, a1\}]}, \quad (12)$$

we obtain

$$\left\{\frac{\mathbf{1}}{\mathbf{4}}, \{\mathbf{a0} \rightarrow -\frac{\mathbf{1}}{\mathbf{2}}, \mathbf{a1} \rightarrow \mathbf{0}\}\right\}. \quad (13)$$

It is easy to interpret the result. The least deviation is $\sqrt{\frac{1}{4}} = \frac{1}{2}$ and (because of uniqueness) the coefficients of the optimal monic quadratic polynomial are $a_1 = 0$ and $a_0 = -\frac{1}{2}$, so the extremal polynomial is

$$p_2^*(x) = x^2 - \frac{1}{2}. \quad (14)$$

We observe that the extremal polynomial in (14) is exactly the same as O_2 given in (9) and consequently differing from T_2 again only by a scaling factor! The next picture shows the deviation of p_2^* from the constant zero polynomial in **I**:

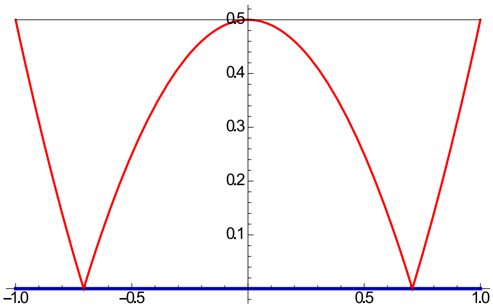

Figure 3: The graph of $|p_2^*(x) - 0|$

If we analyze the syntax of (12), we observe that the problem is a *minimax problem* and this makes the problem difficult from the computational point of view. How does the symbolic optimization algorithm work? One can translate this minimax problem as a real *quantifier elimination* (abbreviated as QE) problem directly as follows (see [6], [13], [14] for the details).
In a nutshell we try give a feel for the algorithmic tool we use in order to deal with extremal polynomials symbolically. We know from calculus that many important properties of a real valued function such as boundedness, continuity, etc., can be expressed via a quantified first order formula containing only 'less' as binary predicate symbol. For instance, the formula

$$\exists_K \forall_x \left|\frac{x}{x^2+1}\right| \leq K \tag{15}$$

evaluates to true or we say that its logical value is true, because the rational function $R(x) = \frac{x}{x^2+1}$ is bounded on the whole real line. One can prove that in elementary algebra, we can always decide the validity of a similar simple formula such as (15) containing only polynomial inequalities, effectively by an algorithm. In addition, if not all variables are bound by quantifiers in the formula, then we cannot say that the formula is true or false, but we can compute a quantifier free formula containing only the free variables which is equivalent to the initial formula, [1], [5] . If we consider formula (15) again, but now without the outermost existential quantifier, the quantifier free equivalent formula is

$$K \geq 1/2, \tag{16}$$

and can be interpreted as an inequality condition on the free variable K which describes the set of possible bounds for the rational function R over the reals.
Now we turn back to Problem 3 for $n = 2$ and solve it with QE. Assume that we represent the monic quadratic extremal polynomial by the coefficient list $(a_0, a_1, 1)$, i.e., $p = a_0 + a_1x + x^2$ is a polynomial which deviates least from zero. We compare it with other monic quadratics, say, with $q = b_0 + b_1x + x^2$. The matrix of the input formula will contain the two monic quadratic polynomials p and q. The next quantified formula expresses in the formal language that p is a polynomial with least deviation.

$$\underset{\substack{x,b_0,b_1 \\ -1\leq x\leq 1}}{\forall} \quad \underset{\substack{y \\ -1\leq y\leq 1}}{\exists} \left((x^2 + a_1x + a_0)^2 \leq (y^2 + b_1y + b_0)^2\right) \tag{17}$$

We have a first order formula with four bound (b_0, b_1, x, y) and 2 free (a_0, a_1) variables. Once we eliminated the quantified variables by the quantifier elimination algorithm we get a condition on the free variables, i.e., we get the coefficients of the extremal polynomial immediately. In Mathematica the built-in function **Resolve** does the effective quantifier elimination.

$$\begin{aligned}\mathbf{Resolve[ForAll[\{x, b0, b1\}, -1 \le x \le 1, Exists[y, -1 \le y \le 1,} \\ \mathbf{(x^2 + a1x + a0)^2 \le (y^2 + b1y + b0)^2]], \{a0, a1\}, Reals]}\end{aligned} \tag{18}$$

Calling the QE algorithm, we obtain again

$$\mathbf{a0 == -\frac{1}{2} \wedge a1 == 0}, \tag{19}$$

that is, we again get the polynomial p_2^* in (14). Note that as an additional information we obtained the uniqueness of the extremal polynomial p. Although this direct QE-approach is theoretically simple, from the computational point of view is not efficient. A more promising approach (because of the less variables involved in the QE problem) is to get the the extremal polynomial in two steps. This is due to C. Brown, see [2], [13]. First the minimum of the square of the norm m is computed, and substituting back the computed optimal value, we get the optimal coefficient in a second step:

[In1 :]

$$\begin{aligned}\mathbf{Resolve[Exists[\{a0, a1\},} \\ \mathbf{ForAll[x, -1 \le x \le 1, (x^2 + a1x + a0)^2 \le m]], Reals]}\end{aligned}$$

[Out1 :]

$$\mathbf{m \ge 1/4},$$

[In2 :]

$$\begin{aligned}\mathbf{Resolve[ForAll[x, -1 \le x \le 1,} \\ \mathbf{(x^2 + a1x + a0)^2 \le 1/4], \{a0, a1\}, Reals]}\end{aligned}$$

[Out2 :]

$$\mathbf{a0 == -1/2 \wedge a1 == 0}. \tag{20}$$

This QE-approach will be used also in the remaining problems, that is in Subsection 2.2 and Section 3. In a similar way we can get for the cubic problem

$$p_3^*(x) = x^3 - 3/4x, \tag{21}$$

and finally it is observed that the extremal polynomials p_j^*'s which solve the extremal Problem 3, in principle are same polynomials as the T_j's and O_j's, which were considered in Problem 1 and Problem 2. So the extremal problem in this subsection is also *a possible, although challenging approach for the introduction of classical Chebyshev polynomials*.

Finally we notice that without the built-in optimization or quantifier elimination methods, an elementary calculus approach can be also used for finding p_2^*. For each ordered pair (a_0, a_1), the biggest deviation in $\mathbf{I}$ is shown in the next picture. To the red point with coordinates $(-1/2, 0)$ belongs the smallest function value for G.

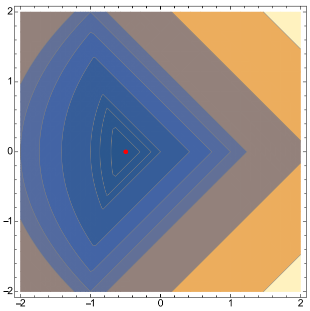

Figure 4: Contour lines of $G(a_0, a_1) = \sup_{x\in\mathbf{I}} |x^2 + a_1x + a_0|$

2.2 Approach 4: Another extremal problem: a point-value Chebyshev problem

Problem 4 *Find a polynomial among the polynomials of degree at most n with real coefficients and with $||p||_\infty \leq 1$ for which the value $|p(x)|$ is maximal for a fixed point x outside the unit interval, i.e, $x \in \mathbf{R}\backslash[-1, 1]$.*

We provide a solution to this problem with QE for $n = 2$. We identify the polynomial $p = a_0 + a_1x + a_2x^2$ of degree at most two by the ordered triplet (a_0, a_1, a_2). The variable B_2 will contain a description of a subset S of the 3D (a_0, a_1, a_2)-space: The points in this set correspond to the polynomials of degree at most two with sup norm bounded by one. The formula describing S can be obtained via QE:

$$\mathbf{B2 = Resolve[ForAll[x, -1 \leq x \leq 1, -1 \leq a0 + a1x + a2x^2 \leq 1], Reals].} \tag{22}$$

Finally among the polynomials described by B_2 we are looking for those for which the value $|p(x)|$ ($|x| > 1$) is maximal. By the QE call

$$\mathbf{Resolve[Abs[x] > 1 \wedge Exists[\{a0, a1, a2\}, B2 \wedge Abs[a0 + a1x + a2x^2] == m], Reals],} \tag{23}$$

we obtain the formula

$$\mathbf{(x < -1 \wedge 0 \leq m \leq -1 + 2x^2) \vee (x > 1 \wedge 0 \leq m \leq -1 + 2x^2)} \tag{24}$$

which clearly indicates that for an arbitrary but fixed x with $|x| > 1$ the maximum is again described by the polynomial $2x^2 - 1$ which is exactly T_2 (see (1))! Thus for $n = 2$, our second extremal problem is solved again by the classical Chebyshev polynomial T_2 (up to sign). Similar investigations for bigger $n > 2$ are led to T_n and left to the interested reader. We close this subsection by noting that the result here can be also interpreted as follows: T_n is the fastest growing polynomial outside $[-1, 1]$ among all polynomials of degree n with $||p||_\infty \leq 1$ (cf. [10, p. 386]).

Figure 5: Contour lines of $|p(x)|$ on the set S.

3 Generalization

For generalization, we focus only on the third approach[2] in this section and consider variations of Problem 3. All will define *new extremal polynomials*. In Problem 3 we can

- alter the interval **I** or replace the interval **I** by a more general set E on the real line. If we treat the unit interval **I** as part of the complex plane **C**, then we can even investigate (complex) extremal polynomials on squares, rectangles, circular arcs and sectors in the complex plane as well, see [8], [7], [11], [12].
- alter the specification of the sought for extremal polynomial: if, in addition to the leading coefficient we fix also the next coefficient of the polynomial, then we can introduce the class of *Zolotarev polynomials*, see [9], [6].
- we can request additional constraints on the monic extremal polynomials p_n: e.g., it should also satisfy that $p_n(-1) = p_n(1) = 0$, see [9].

Here we consider only one particular complex Chebyshev example for $deg = 2, E = A^{\pi/2} = \{z : |z| = 1 \wedge |\arg z| < \frac{\pi}{2}\}$, that is, a monic quadratic extremal polynomial over the complexes on the unit semicircle which is symmetric w.r.t. the real axis is sought [12]. Assume that the sought for extremal polynomial has the form $p_2(z) = z^2 + a_1 z + a_0$, where a_1 and a_0 should be determined. It is clear that because of the symmetry of the circular arc, all the coefficients of the extremal polynomial is real, therefore we will formulate the optimization problem as a real quantifier elimination problem in two different ways. The first expresses the square of the norm of the polynomial p_2 in terms of the real part $\Re(z) = x$. Thus we have

$$\begin{aligned}\mathbf{Resolve[Exists[\{a0, a1\}, ForAll[x, 0 \leq x \leq 1,}\\ \mathbf{1 - 2a0 + a0^2 + a1^2 + 2a1x + 2a0a1x + 4a0x^2 \leq m]], Reals]}\end{aligned} \tag{25}$$

[2]In the literature generalized Chebyshev polynomials may also refer to Shabat polynomials, which own exactly two different two (complex) critical values, but we do not consider them here.

We get $m \geq 4(3-2\sqrt{2})$. We substitute inf of m back to get the extremal coefficients a_0, a_1:

$$\begin{gathered}\mathbf{RootReduce[Resolve[ForAll[x, 0 \leq x \leq 1,} \\ \mathbf{1 - 2a0 + a0^2 + a1^2 + 2a1x + 2a0a1x + 4a0x^2 \leq 4(3-2\sqrt{2})],} \\ \mathbf{Reals, Backsubstitution \to True]]}\end{gathered} \tag{26}$$

We obtain $a_1 = \sqrt{2}-2$ and $a_0 = \sqrt{2}-1$, thus the quadratic (monic) Chebyshev polynomial on this unit-semicircle is

$$p_2^*(z) = T_2^{A^{\pi/2}} = z^2 + (\sqrt{2}-2)z + (\sqrt{2}-1). \tag{27}$$

The second uses the fact that the map $z : x \to \frac{x-i}{x+i}$ maps $\mathbf{R}\backslash\mathbf{I}$ to the complex semicircle $A^{\pi/2}$. Thus in terms of x we have the following extremal problem expressed as a QE problem.

$$\begin{gathered}\mathbf{Resolve[Exists[\{a0, a1\}, ForAll[x, Abs[x] >= 1,} \\ \mathbf{(1 + 2a0 + a0^2 - 2a1 - 2a0a1 + a1^2 + 2x^2 - 12a0x^2 +} \\ \mathbf{2a0^2x^2 + 2a1^2x^2 + x^4 + 2a0x^4 + a0^2x^4 + 2a1x^4 +} \\ \mathbf{2a0a1x^4 + a1^2x^4)/(x^2+1)^2 \leq m]], Reals]}\end{gathered} \tag{28}$$

Again, we get $m \geq 4(3-2\sqrt{2})$ and consequently $a_1 = \sqrt{2}-2$ and $a_0 = \sqrt{2}-1$, as above. Finally we note that the extremal polynomial can be also characterized by its roots in complex plane, i.e.

$$p_2^*(z) = (z - z_1)(z - \overline{z_1}), \tag{29}$$

where $z_1 = \frac{1}{2}(2 - \sqrt{2} - i\sqrt{8\sqrt{2}-10}) \approx 0.2929 - 0.5731i$. If we compute quadratic Chebyshev polynomials not only for the semicircle $A^{\pi/2}$, but for a unit circular arc which is symmetric w.r.t. the real axis where the half-opening angle α of the arc A^α varies between 0 and π, and visualize the root orbits of the infinitely many associated extremal polynomials on the complex plane, we get two curves starting from 1 and ending up at 0. Thus the two extremal polynomials which belong to $\alpha = 0$ and $\alpha = \pi$ (in fact already for any α which is between $\frac{2\pi}{3}$ and π) are $(z-1)^2$ and z^2, respectively.

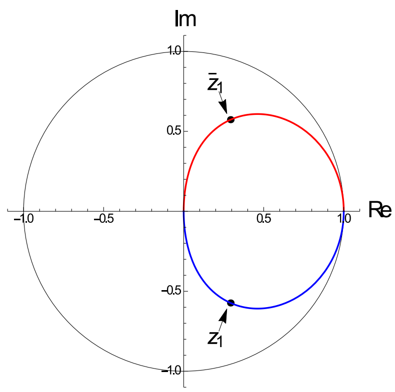

Figure 6: Root orbits of quadratic Chebyshev polynomials on circular arcs

4 Conclusion

We shed light to four exploration paths following different problems. In the classical case they all lead to same polynomial family: to the classical Chebyshev polynomials T_n. We put an emphasis on extremality. Chebyshev polynomials can be also introduced as a solution of a particular extremal problem as we saw in Section 2.1 and 2.2. We demonstrated that computer algebra systems equipped with symbolic algorithms may help to consider, to compute, and to introduce extremal polynomials constructively. We solved some extremal problems via the built-in optimization tools of Mathematica (a black-box approach) and also with the real quantifier elimination algorithm (as a white-box approach).

In this short paper we just touched the possible problems and the corresponding algorithms and tools for the symbolic computer-aided treatment of the problems and do not investigated the limits of the current implementations and possible numerical approaches and the connections between the complex and real generalized extremal problems. As the degree of the sought for extremal polynomial increases, some of the computational problems are unsolvable symbolically. Then we have to exploit additional knowledge or the solution may be reconstructed from (a high enough precision) numerical approximation of the extremal polynomial. Unfortunately for the generalized extremal problems, especially in the complex plane, a nice, complete closed-form or recursive solution is known (or can be expected) only in a very few cases.

The author develops in the frame of a Hungarian-Serbian IPA project "Non-Standard Forms of Teaching Mathematics and Physics: Experimental and Modeling Approach" a whole course material for the (generalized) extremal polynomials introduced in Section 3 and does research on the symbolic computation of extremal polynomials. A GeoGebra course material development along this line were also produced with Zoltán Kovács, see for instance the interactive material on the GeoGebra Tube `http://www.geogebratube.org/student/mFK8kqDa5`.

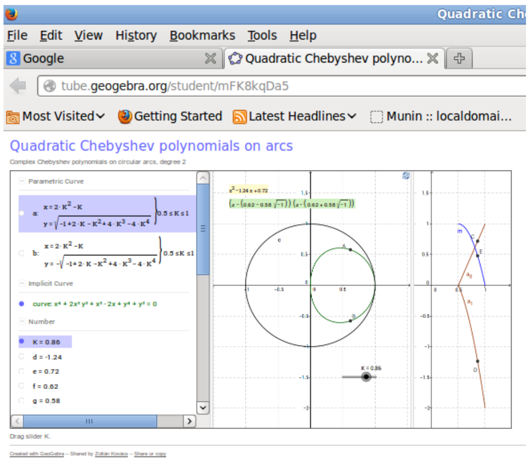

Figure 7: GeoGebra spreadsheet for quadratic Chebyshev polynomials on circular arcs

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